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Thus the degree each time is half what it was before. If $n-m$, $m-m'$, $m'-m''$, etc., be *even*, we shall have for the successive degees,

$$\begin{aligned}
&\text{1st, } 2m+1+2n+1 = 2(m+n+1),\\
&\text{2nd, } 2m+1+2m'+1 = 2(m'+m+1) = m+n,\\
&\text{3rd, } 2m'+1+2m''+1 = 2(m''+m'+1) = \tfrac{1}{2}(m+n-2),\\
&\text{4th, } 2m''+1+2m'''+1 = 2(m'''+m''+1) = \tfrac{1}{4}(m+n-2), \text{ etc.}
\end{aligned}$$

After the second, the degree is each time half the preceding. If the exponents $n-m$, $m-m'$, etc., were some even and some odd the result would be between these two.

A numerical example is added.—Let $m=9$ and $n=14$. Then

$$\begin{aligned}
\cos^{19}\varphi \sin^{29}\varphi &= (\tfrac{1}{2})^{24}(\sin 2\varphi)^{19}[1-\cos 2\varphi]^5,\\
(\cos 2\varphi)^5(\sin 2\varphi)^{19} &= (\tfrac{1}{2})^{12}(\sin 4\varphi)^5[1-\cos 4\varphi]^7,\\
(\cos 4\varphi)^7(\sin 4\varphi)^5 &= (\tfrac{1}{2})^{6}(\sin 8\varphi)^5[1+\cos 8\varphi],\\
\cos 8\varphi\,(\sin 8\varphi)^5 &= (\tfrac{1}{2})^{3}(\sin 16\varphi)[1-\cos 16\varphi]^2,\\
\cos 16\varphi \sin 16\varphi &= \tfrac{1}{2}\sin 32\varphi.
\end{aligned}$$

If in the expansion of the right-hand member of the first equation we take $(\cos 2\varphi)^3$ instead of $(\cos 2\varphi)^5$ we have

$$\begin{aligned}
(\cos 2\varphi)^3(\sin 2\varphi)^{19} &= (\tfrac{1}{2})^{11}(\sin 4\varphi)^3[1-\cos 4\varphi]^8,\\
(\cos 4\varphi)^7(\sin 4\varphi)^3 &= (\tfrac{1}{2})^{5}(\sin 8\varphi)^3[1+\cos 8\varphi]^2,\\
\cos 8\varphi\,(\sin 8\varphi)^3 &= (\tfrac{1}{2})^{2}(\sin 16\varphi)[1-\cos 16\varphi],\\
&= (\tfrac{1}{2})^{2}[\sin 16\varphi - \tfrac{1}{2}\sin 32\varphi].
\end{aligned}$$

DISCUSSION OF AN EQUATION.

BY JOHN BORDEN, CHICAGO, ILL.

As equations of the fifth or less degree are reducible to the form

$$y^n + py + q = 0, \tag{1}$$

n being an integer and positive, its discussion is of interest.

QUESTION.—Determine the number of real roots, their limits, and sign, for all real values of p and q.

ANSWER:—I. *If* n *is odd and* p *is positive,* there is only one real root, whose sign is contrary to that of q and whose limits are $q \div p \times -0$, and $q \div p \times -1$.

II. *If* n *is odd and* p *is negative,* there are three real roots when

$$\frac{p^n}{q^{n-1}} \overset{=}{>} \frac{n^n}{(n-1)^{n-1}},$$

two of which have the same sign as q, and become equal to each other, and of the arithmetical value of $n\div(n-1)$, when $p^n\div q^{n-1} = n^n\div(n - 1)^{n-1}$, and become imaginary for all values of p and q which make $p^n\div q^{n-1} < n^n\div(n-1)^{n-1}$. Their limits are, for one, $q\div p\times-n\div(n-1)$, and $q\div p\times-1$; and for the other, $q\div p\times-n\div(n-1)$, and $q\div p\times-\infty$. The other real root remains real for all values of p and q and its sign is contrary to that of q, and its limits are $q\div p\times+0$, and $q\div p\times+\infty$.

III. *If* n *is even and* q *is positive,* there are two real roots when

$$\frac{p^n}{q^{n-1}} \overset{=}{>} \frac{n^n}{(n-1)^{n-1}},$$

both of the same sign and which is contrary to that of p. They become equal and also of the arithmetical value of $n\div(n-1)$, when $p^n\div q^{n-1} = n^n\div(n-1)^{n-1}$; and they become imaginary for all values of p and q which make $p^n\div q^{n-1} < n^n\div(n-1)^{n-1}$. Their limits are, for one, $q\div p\times-n\div(n-1)$ and $q\div p\times-1$, and for the other, $q\div p\times-n\div(n-1)$ and $q\div p\times-\infty$.

IV. *If* n *is even and* q *is negative,* there are always two real roots having contrary signs. Their limits are, for one, $q\div p\times-0$ and $q\div p\times-1$, and for the other, $q\div p\times+0$ and $q\div p\times+\infty$.

V. Eq. (1) has no other real roots than as above.

PROOF:—In Eq. (1) make $y = \frac{q}{p}.x.$ (2)

Then
$$x^n + \frac{p^n}{q^{n-1}}.x + \frac{p^n}{q^{n-1}} = 0; \tag{3}$$

which gives
$$\frac{p^n}{q^{n-1}} = -\frac{x^n}{x+1}. \tag{4}$$

If n *is odd, the sign of the first term of Eq.* (4) *depends on* p.

Suppose p *is positive,* then no positive value of x can satisfy Eq. (4). Therefore x must be negative. In such case (4) assumes the arithm'l form

$$\frac{p^n}{q^{n-1}} = \frac{x^n}{1-x}. \tag{5}$$

Now any value of x greater than one is evidently impossible. For $x = 1$, $p^n\div q^{n-1} = \infty$. For $x = 0$, $p^n\div q^{n-1} = 0$. Therefore in Eq. (3), in this case, the only real value of x is negative and its limits are -0, and -1.

Therefore in this case the only real value of y in Eq. (1) has for its limits $q\div p\times-0$, and $q\div p\times-1$. As p is supposed to be positive, it is evident that the sign of the real values of y is contrary to that of q. This proves I.

Next suppose p *is negative*—n still remaining odd. Then for a positive value of x, Eq. (4) assumes the arithmetical form,

$$\frac{p^n}{q^{n-1}} = \frac{x^n}{x+1}, \tag{6}$$

and for a negative value of x, Eq. (4) assumes the arithmetical form

$$\frac{p^n}{q^{n-1}}=\frac{x^n}{x-1}. \tag{7}$$

Now, in Eq. (6), for $x=0$, $p^n\div q^{n-1}=0$, and for $x=\infty$, $p^n\div q^{n-1}=\infty$. Therefore the corresponding values of y in Eq. (1) are $q\div p\times+0$ and $q\div p\times+\infty$ As p is here assumed to be negative this value of y will have a sign contrary to that of q. This proves one branch of Prop. II.

In Eq. (7) x can never be less than unity, because, for $x=1$, $p^n\div q^{n-1}=\infty$; also, for $x=\infty$, $p^n\div q^{n-1}=\infty$. Now it will be found, on trial, that as the value of x increases above unity the value of $p^n\div q^{n-1}$ decreases, also, that as the value of x decreases from ∞ the value of $p^n\div q^{n-1}$ decreases. Here then are two values of x approaching each other, and it will be found that $p^n\div q^{n-1}$ is a minimum when $=n^n\div(n-1)^{n-1}$, and that, in such case, the two values of x have become each equal to $n\div(n-1)$. At this point these two real roots vanish and become imaginary. Apply the rule for maxima and minima and this will appear. Therefore in Eq. (7) there are two values for x, one lying between 1 and $n\div(n-1)$ and the other between $n\div(n-1)$ and ∞; and these two negative values for x, in the present case, give as the corresponding values of y; one, $q\div p\times-n\div(n-1)$ and $q\div p\times-1$, the other $q\div p\times-n\div(n-1)$ and $q\div p\times-\infty$. As p is here supposed to be negative the signs of these two values of y are the same as that of q. Thus is proved the other branch of Prop. II.

To illustrate; in $y^3+py+q=0$, it is well known that there is only one real root when p is positive, and that, when p is negative, there are three real roots when $p^3\div q^2$ is equal or greater than $3^3\div2^2$ or $\frac{27}{4}$.

So also in $y^5+py+q=0$, it will be found as above that when p is positive there is only one real root, and that when p is negative there are three real roots when $p^5\div q^4=$ or $>5^5\div4^4$. When $p^5\div q^4=5^5\div4^4$, then two of the values of y are equal, such values being $y=q\div p\times-5\div4$.

If n *is even, the sign of the first term of Eq.* (4) *depends on* q. *Suppose* q *is positive.* Then no positive value of x can satisfy Eq. (4). But for x negative Eq. (4) reduces to the arithmetical form

$$\frac{p^n}{q^{n-1}}=\frac{x^n}{x-1}. \tag{8}$$

This is similar in form to Eq. (7), and in like manner proves Prop. III.

Suppose q *is negative—n* still remaining even. Then for x positve Eq. (4) assumes the arithmetical form

$$\frac{p^n}{q^n}=\frac{x^n}{x+1}. \tag{9}$$

Also for x negative Eq. (4) assumes the arithmetical form

$$\frac{p^n}{q^n}=\frac{x^n}{1-x}. \qquad (10)$$

The value of x in Eq. (9) can lie any where between 0 and infinity; but the value of x in Eq. (10) can only lie between 0 and 1. This proves IV. And Prop. V. also follows from the foregoing.

Note. The cases in which n is fractional, and, in which p and q (both or either) are imaginary, have not, though they might have, been considered.

Also, an equation of the form, $z^n+p'z^{n-1}+q'=0$ is merely the reciprocal form of Eq. (1). For if $z=\frac{1}{y}$, then $y^n+\frac{p'}{q'}y+\frac{1}{q'}=0$, which is the same as Eq. (1).

THE CENTRE OF GRAVITY OF THE APPARENT DISK OF A PLANET.

BY PROF. A. HALL, U. S. NAVAL OBSERVATORY, WASHINGTON, D. C.

When a planet is observed from the Earth its disk will not be completely illuminated except at the time of opposition. At other times it will appear gibbous or crescent, If we assume the planet to be a sphere, which, according to the best observations is the case with Mercury, Venus and Mars, the apparent disk will be composed of a semi-circle and a semi-ellipse which are separated by the line of cusps. If a right line be drawn through the centre of the line of cusps and at right angles to it, the centre of gravity of the apparent disk is in this line. In the plane triangle between the Sun, the Earth and the planet let φ be the angle at the planet; let a be the radius of the semi-circular part of the disk, and designate by m the distance of the centre of gravity from the centre of the line of cusps. Then when the planet is gibbous we shall have

$$m=\frac{8a}{3\pi}\sin\tfrac{1}{2}\varphi^2;$$

and when the planet is crescent,

$$m=\frac{8a}{3\pi}\cos\tfrac{1}{2}\varphi^2,$$

where π is the number 3.1416. The angle φ and the radius a can be computed from the data given in the ephemerides, and the distance m is known therefore for any time.